Non-classical conditional probability and the quantum no-cloning theorem

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The quantum mechanical no-cloning theorem for pure states is generalized and transfered to the quantum logics with a conditional probability calculus. This is, on the one hand, an extension of the classical probability calculus and, on the other hand, a mathematical generalization of the Lüders - von Neumann quantum measurement process. In the non-classical case, a very special type of conditional probability emerges, describing the probability for the transition from a past event to a future event independently of any underlying state. This probability results from the algebraic structure of the quantum logic only and is invariant under algebraic morphisms, which is used to prove the generalized no-cloning theorem in a rather abstract, though simple and basic fashion without relying on a tensor product construction or finite dimension as required in other approaches.

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I. INTRODUCTION

A pioneering result with far-reaching consequences in quantum information and communication theory is the no-cloning theorem, stating that unknown pure quantum states cannot be copied unless they are orthogonal [12, 29–31]. An interesting generalization is the no-broadcasting theorem for mixed states [3]. Originally, both were proved in Hilbert space quantum mechanics, then extended to the C*-algebraic setting [11] and later to finite-dimensional generic probabilistic models [1, 2] and to quantum logics [20]. In the latter case, only universal cloning is impossible, while the cloning of a small set or pair of states can be ruled out in the other cases. Though these results preclude the perfect cloning, the approximate or imperfect cloning of quantum states remains possible [8, 10, 19]. In this paper, the (perfect) cloning of a small set or pair of states is considered in the setting of quantum logics with a conditional probability calculus [21, 22], including finite-dimensional as well as infinite-dimensional models.

A quantum logic E is a purely algebraic structure for the quantum events (or propositions). It is quite common to use an orthomodular partially ordered set or lattice [4, 5, 17, 25]. States are then defined in the same way as the classical probability measures, and conditional probabilities are postulated to behave like the classical ones on compatible subsets of E. Note that a subset is called compatible if it is contained in another subset of E forming a Boolean algebra (i.e., in a classical subsystem of E) [7]. Some quantum logics entail unique conditional probabilities, many others don't. The classical Boolean algebras and the Hilbert space quantum logic (consisting of the closed subspaces or, equivalently, the selfadjoint projection operators) do and, in the latter case, conditionalization becomes identical with the state transition of the Lüders - von Neumann (i.e., projective) quantum measurement process [21]. Therefore, the quantum logics with unique conditional probabilities can be regarded as a generalized mathematical model of projective quantum measurement.

In this framework, a very special type of conditional probability emerges in the non-classical case [21, 22]. It describes the probability for the transition from a past event e to a future event f, independently of any underlying state, and results from the algebraic structure of the quantum logic E. This probability exists only for certain event pairs e and f. It exists for all events $f \in E$, if e is a minimal event (*atom*) in E. The states resulting in this way are called *atomic*. They represent a generalization of the pure states in Hilbert space quantum mechanics.

After the early pioneering work by Birkhoff and von Neumann in 1937 [6], quantum logics have been studied extensively between 1960 and 1995 [4, 5, 17, 18, 24–28]. Various forms of conditional probability have also been considered [4, 9, 13–16]. However, the quantum logics which possess unique conditional probabilities and particularly the special type of the state-independent conditional probability have not attracted any attention before the author's work [21, 22].

Considering such a quantum logic, this special type of conditional probability is used in the present paper to prove, in a very basic fashion, the generalized no-cloning theorem for atomic states. A tensor product construction as used in the other approaches is not required. Instead, the embedding of two copies of E, which shall be compatible with each other, in a larger quantum logic L is sufficient.

The paper is organized as follows. The algebraic structure of the quantum logic is considered in section II. Section III then turns to states and briefly sketches the non-classical conditional probability calculus from Refs. [21, 22]. The main results are presented in sections IV and V. The proof of the quantum mechanical no-cloning

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theorem rests upon the following two basic properties of the inner product in the Hilbert space: It is invariant under the unitary cloning transformation and multiplicative for the Hilbert space tensor product. These properties are transferred to the non-classical conditional probabilities in a certain way (Lemmas 1(a) and 2), which then allows to mimic the quantum mechanical proof for the generalized no-cloning theorem (Theorem 1). In section VI, the ties to Hilbert space quantum mechanics are pointed out.

II. COMPATIBILITY IN ORTHOMODULAR PARTIALLY ORDERED SETS

In quantum mechanics, the measurable quantities of a physical system are represented by observables. Most simple are those observables where only the two discrete values 0 and 1 are possible as measurement outcome; these observables are called *events* (or *propositions*) and are elements of a mathematical structure called *quantum logic*.

In this paper, a quantum logic shall be an orthomodular partially ordered set E with the partial ordering $\leq$, the orthocomplementation $'$, the smallest element 0 and the largest element $\mathbb{I}$ [4, 5, 17, 25]. This means that the following conditions are satisfied by all $e, f \in E$:

(A) $e \leq f$ implies $f' \leq e'$.
(B) $(e')' = e$.
(C) $e \leq f'$ implies $e \vee f$, the supremum of e and f, exists.
(D) $e \vee e' = \mathbb{I}$.
(E) $f \leq e$ implies $e = f \vee (e \wedge f')$. (orthomodular law)

Here, $e \wedge f$ denotes the infimum of e and f, which exists iff $e' \vee f'$ exists. Two elements $e, f \in E$ are called *orthogonal* if $e \leq f'$ or, equivalently, $f \leq e'$. An element $e \neq 0$ in E is called an *atom* if there is no element f in E with $f \leq e$ and $0 \neq f \neq e$.

The interpretation of this mathematical terminology is as follows: orthogonal events are exclusive, e' is the negation of e, and $e \vee f$ is the disjunction of the two exclusive events e and f.

It is not assumed that E is a *lattice* (in a lattice, there is a smallest upper bound $e \vee f$ and largest lower bound $e \wedge f$ for any two elements e and f). If E were a distributive lattice (i.e., $e \wedge (f \vee g) = (e \wedge f) \vee (e \wedge g)$ for all $e, f, g \in E$), it would become a Boolean lattice or Boolean algebra. The orthomodular law is a weakening of the distributivity law.

Classical probability theory uses Boolean lattices as mathematical structure for the random events, and it can be expected that those subsets of E, which are Boolean lattices, behave classically. Therefore, a subset E_0 of E is called *compatible* if there is a Boolean lattice B with $E_0 \subseteq B \subseteq E$. Any subset with pairwise orthogonal element is compatible [7]. Two subsets E_1 and E_2 of E are called *compatible with each other* if the union of any compatible subset of E_1 with any compatible subset of E_2 is a compatible subset of E. Note that this does not imply that E_1 or E_2 themselves are compatible subsets.

A subset of an orthomodular lattice is compatible if each pair of elements in this subset forms a compatible subset. However, the pairwise compatibility of the elements of a subset of an orthomodular partially ordered set does not any more imply the compatibility of this subset [7].

A quantum logical structure, which is more general than the orthomodular partially ordered sets, has been used in Refs. [21, 22]. This more general structure is sufficient when only compatible pairs of elements in the quantum logic are considered. However, compatible subsets with more than two elements will play an important role in this paper.

A quantum logic is a purely algebraic structure, unfurling its full potential only when its state space has some nice properties which shall be considered in the next section.

III. NON-CLASSICAL CONDITIONAL PROBABILITY

The states on the orthomodular partially ordered set E are the analogue of the probability measures in classical probability theory, and conditional probabilities can be defined similar to their classical prototype. A *state* ρ allocates the probability $\rho(f)$ with $0 \leq \rho(f) \leq 1$ to each event $f \in E$, is additive for orthogonal events, and $\rho(\mathbb{I}) = 1$. It then follows that $\rho(f) \leq \rho(e)$ for any two events $e, f \in E$ with $f \leq e$.

The *conditional probability* of an event f under another event e is the updated probability for $f \in E$ after the outcome of a first measurement has been the event $e \in E$; it is denoted by $\rho(f|e)$. Mathematically, it is defined by the conditions that the map $E \ni f \to \rho(f|e)$ is a state on E and that it coincides with the classical conditional probability for those f which are compatible with e. The second condition is equivalent to the identity $\rho(f|e) = \rho(f)/\rho(e)$ for all events $f \in E$ with $f \leq e$. It must be assumed that $\rho(e) \neq 0$.

However, among the orthomodular partially ordered sets, there are many where no states or no conditional probabilities exist, or where the conditional probabilities are ambiguous. It shall now be assumed for the remaining part of this paper that

(F) there is a state ρ on E with $\rho(e) \neq 0$ for each $e \in E$ with $e \neq 0$,

(G) E possesses unique conditional probabilities, and

(H) the state space of E is strong; i.e., if
$$\{\rho \mid \rho \text{ is a state with } \rho(f) = 1\} \subseteq \{\rho \mid \rho \text{ is a state with } \rho(e) = 1\}$$
holds for two events e and f in E, then $f \leq e$.

If ρ is a state with $\rho(e) = 1$ for some event $e \in E$, then $\rho(f|e) = \rho(f)$ for all $f \in E$. This follows from (G).

For some event pairs e and f in E, the conditional probability does not depend on the underlying state; this means $\rho_1(f|e) = \rho_2(f|e)$ for all states ρ_1 and ρ_2 with $\rho_1(e) \neq 0 \neq \rho_2(e)$. This special conditional probability is then denoted by $\mathbb{P}(f|e)$. The following two conditions are equivalent for an event pair $e, f \in E$:
(i) $\mathbb{P}(f|e)$ exists and $\mathbb{P}(f|e) = s$.
(ii) $\rho(e) = 1$ implies $\rho(f) = s$ for the states ρ on E.
Due to condition (H), $f \leq e$ holds for two events e and f in E if and only if $\mathbb{P}(e|f) = 1$. Moreover, e and f are orthogonal if and only if $\mathbb{P}(e|f) = 0$.

$\mathbb{P}(f|e)$ exists for all $f \in E$ if and only if e is an *atom* (minimal event), which results in the atomic state $\mathbb{P}_e$ defined by $\mathbb{P}_e(f) := \mathbb{P}(f|e)$. This is the unique state allocating the probability value 1 to the atom e. For two atoms e and f in E, the following four identities are equivalent: $\mathbb{P}_e(f) = 1$, $\mathbb{P}_f(e) = 1$, $\mathbb{P}_e = \mathbb{P}_f$, and $e = f$.

IV. MORPHISMS

In this section, the invariance of the special conditional probability $\mathbb{P}(\cdot|\cdot)$ under quantum logical morphisms is studied. In the proof of the main result, this will later replace the invariance of the inner product under unitary transformations in the Hilbert space setting.

Suppose E and F are orthomodular partially ordered sets and $T : E \to F$ is an (algebraic) morphism (i.e., $Te_1 \leq Te_2$ for $e_1, e_2 \in E$ with $e_1 \leq e_2$, $T(e') = (Te)'$ for all $e \in E$ and $T\mathbb{I} = \mathbb{I}$). A dual transformation T^*, mapping the states ρ on F to states $T^*\rho$ on E, is then defined by $(T^*\rho)(e) := \rho(Te)$ for $e \in E$. In the case where both E and F possess unique conditional probabilities,

$$(T^*\rho)(e_2|e_1) = \rho(Te_2|Te_1)$$

holds for all events $e_1, e_2 \in E$ with $\rho(Te_1) \neq 0$. To see this, consider the state $e \to \rho(Te|Te_1)$ on E; the uniqueness of the conditional probability implies that it must coincide with the state $e \to (T^*\rho)(e|e_1)$.

Lemma 1: Let E and F be orthomodular partially ordered sets, satisfying (F) and (G), and let $T : E \to F$ be a morphism.
(a) If $\mathbb{P}(e_2|e_1)$ exists for two events e_1 and e_2 in E with $Te_1 \neq 0$, then $\mathbb{P}(Te_2|Te_1)$ exists and

$$\mathbb{P}(Te_2|Te_1) = \mathbb{P}(e_2|e_1).$$

(b) $T^*\mathbb{P}_f = \mathbb{P}_{T^{-1}f}$ for the atoms f in F.

Proof. (a) Suppose $\mathbb{P}(e_2|e_1)$ exists for e_1 and e_2 in E. Then $\mathbb{P}(e_2|e_1) = (T^*\rho)(e_2|e_1) = \rho(Te_2|Te_1)$ for all states ρ on F with $(T^*\rho)(e_1) = \rho(Te_1) > 0$. Therefore, $\mathbb{P}(Te_2|Te_1)$ exists and is identical with $\mathbb{P}(e_2|e_1)$.
(b) Let f be an atom in F. Then $T^{-1}f$ is an atom in E and, by (a), $\mathbb{P}(e|T^{-1}f) = \mathbb{P}(Te|f)$ for $e \in E$. Therefore, $\mathbb{P}_{T^{-1}f} = T^*\mathbb{P}_f$.

V. THE GENERALIZED NO-CLONING THEOREM

In this section, a quantum logic shall always be an orthomodular partially ordered set and shall satisfy (F), (G) and (H). Suppose that E is a quantum logic and that two copies of it are contained in the larger quantum logic L. This means that there are two injective morphisms $\pi_1 : E \to L$ and $\pi_2 : E \to L$. Moreover, suppose

(I) the subsets $\pi_1(E)$ and $\pi_2(E)$ of L are compatible with each other and

(J) $\pi_1(e) \wedge \pi_2(f)$ is an atom in L for each pair of atoms e and f in E.

The proof of the quantum mechanical no-cloning theorem rests upon the multiplicativity of the inner product for the Hilbert space tensor product. The following lemma provides the substitute for this in the more general setting.

Lemma 2: Suppose $\mathbb{P}(e_2|e_1)$ and $\mathbb{P}(f_2|f_1)$ both exist for $e_1, e_2, f_1, f_2 \in E$. Then

$$\mathbb{P}((\pi_1 e_2) \wedge (\pi_2 f_2)|(\pi_1 e_1) \wedge (\pi_2 f_1))$$

exists and

$$\mathbb{P}((\pi_1 e_2) \wedge (\pi_2 f_2)|(\pi_1 e_1) \wedge (\pi_2 f_1)) = \mathbb{P}(e_2|e_1)\mathbb{P}(f_2|f_1).$$

Proof. Given the above assumptions, consider a state ρ on L with

$$\rho((\pi_1 e_1) \wedge (\pi_2 f_1)) = 1.$$

Then $\rho(\pi_1 e_1) = 1 = \rho(\pi_2 f_1)$. Now define two states μ_1 and μ_2 on E by

$$\mu_1(e) := \rho((\pi_1 e) \wedge (\pi_2 f_1)) \text{ and } \mu_2(e) := \rho((\pi_1 e))$$

for $e \in E$. Note that μ_1 is a state due to (I). Since $\mu_1(e_1) = 1 = \mu_2(e_1)$ and $\mathbb{P}(e_2|e_1)$ exists, it follows that $\mathbb{P}(e_2|e_1) = \mu_1(e_2) = \mu_2(e_2)$ and $\mathbb{P}(e_2'|e_1) = \mu_1(e_2') = \mu_2(e_2')$. Thus

$$\mathbb{P}(e_2|e_1) = \rho((\pi_1 e_2) \wedge (\pi_2 f_1)) = \rho((\pi_1 e_2))$$

and

$$\mathbb{P}(e_2'|e_1) = \rho((\pi_1 e_2') \wedge (\pi_2 f_1)) = \rho((\pi_1 e_2')).$$

In the case $\mathbb{P}(e_2|e_1) > 0$, define the state ν on E by

$$\nu(f) := \frac{\rho((\pi_1 e_2) \wedge (\pi_2 f))}{\mathbb{P}(e_2|e_1)}$$

for $f \in E$. Then $\nu(f_1) = 1$ and therefore

$$\mathbb{P}(f_2|f_1) = \nu(f_2) = \frac{\rho((\pi_1 e_2) \wedge (\pi_2 f_2))}{\mathbb{P}(e_2|e_1)}.$$

In the case $\mathbb{P}(e_2|e_1) = 0$, it follows $\mathbb{P}(e_2{}'|e_1) = 1$. Then $e_1 \le e_2{}'$ and $e_2 \le e_1{}'$. Therefore,

$$\rho((\pi_1 e_2)\wedge(\pi_2 f_2)) \le \rho(\pi_1 e_2) \le \rho(\pi_1 e_1{}') = 1-\rho(\pi_1 e_1) = 0$$

and $\rho((\pi_1 e_2) \wedge (\pi_2 f_2)) = 0$. In both cases,

$$\rho((\pi_1 e_2) \wedge (\pi_2 f_2)) = \mathbb{P}(e_2|e_1)\mathbb{P}(f_2|f_1).$$

Since this holds for all states ρ on L with $\rho((\pi_1 e_1) \wedge (\pi_2 f_1)) = 1$, it finally follows that

$$\mathbb{P}((\pi_1 e_2) \wedge (\pi_2 f_2)|(\pi_1 e_1) \wedge (\pi_2 f_1)) = \mathbb{P}(e_2|e_1)\mathbb{P}(f_2|f_1).$$

Note that the proof of Lemma 2 does neither require assumption (J) nor any tensor product construction, but instead only assumption(I).

A state ρ on L can be restricted to each one of the two copies of E in L, resulting in the following two states on E: $\pi_1^*\rho = \rho\pi_1$ and $\pi_2^*\rho = \rho\pi_2$.

Lemma 3: Let e and f be atoms in E and ρ a state on L. Then $\rho\pi_1 = \mathbb{P}_e$ and $\rho\pi_2 = \mathbb{P}_f$ if and only if $\rho = \mathbb{P}_{(\pi_1 e)\wedge(\pi_2 f)}$.

Proof. Assume $\rho\pi_1 = \mathbb{P}_e$ and $\rho\pi_2 = \mathbb{P}_f$. Then (I) implies

$$1 = \mathbb{P}_e e = \rho\pi_1 e = \rho\left((\pi_1 e) \wedge (\pi_2 f)\right) + \rho\left((\pi_1 e) \wedge (\pi_2 f')\right)$$

and

$$0 \le \rho\left((\pi_1 e) \wedge (\pi_2 f')\right) \le \rho\pi_2 f' = \mathbb{P}_f f' = 0.$$

Therefore $1 = \rho\left((\pi_1 e) \wedge (\pi_2 f)\right)$ and, since $(\pi_1 e) \wedge (\pi_2 f)$ is an atom in L,

$$\rho = \mathbb{P}_{(\pi_1 e)\wedge(\pi_2 f)}.$$

Now assume $\rho = \mathbb{P}_{(\pi_1 e)\wedge(\pi_2 f)}$ and $a \in E$. Then by Lemmas 1(a) and 2

$$\begin{aligned}\rho\pi_1 a &= \mathbb{P}((\pi_1 a)|(\pi_1 e) \wedge (\pi_2 f))\\ &= \mathbb{P}((\pi_1 a) \wedge (\pi_2 \mathbb{I})|(\pi_1 e) \wedge (\pi_2 f))\\ &= \mathbb{P}(\pi_1 a|\pi_1 e)\mathbb{P}(\pi_2 \mathbb{I}|\pi_2 f)\\ &= \mathbb{P}(\pi_1 a|\pi_1 e)\\ &= \mathbb{P}(a|e) = \mathbb{P}_e a\end{aligned}$$

The second identity $\rho\pi_2 = \mathbb{P}_f$ follows in the same way.

Now suppose that C is a set of atoms in E and that f is a fixed atom in E, that the local state on the first copy of E is any element in $\{\mathbb{P}_e|e \in C\}$ and that the local state on the second copy of E is $\mathbb{P}_f$. For the state ρ on L this means that $\rho\pi_1 = \mathbb{P}_e$ for some unknown $e \in C$ and $\rho\pi_2 = \mathbb{P}_f$.

In the usual quantum mechanical setting, the cloning is performed by a unitary transformation on the Hilbert space tensor product. In this paper, it shall be performed by an automorphism of L; cloning means that the automorphism transforms the initial local state on the second copy of E to a copy of the unchanged local state on the first copy of E. After the transformation, both copies of E are in the same local state and this is the local state on the first copy before the transformation.

Definition 1: A cloning transformation for $\{\mathbb{P}_e|e \in C\}$ is an automorphism T of the quantum logic L such that $(T^*\rho)\pi_1 = \rho\pi_1 = (T^*\rho)\pi_2$ holds for the states ρ on L with $\rho\pi_1 \in \{\mathbb{P}_e|e \in C\}$ and $\rho\pi_2 = \mathbb{P}_f$.

Theorem 1: A cloning transformation T for $\{\mathbb{P}_e|e \in C\}$ exists only if the atoms in C are pairwise orthogonal.

Proof. Assume T is a cloning transformation for $\{\mathbb{P}_e|e \in C\}$. Note that, by Lemma 3, $\rho\pi_1 = \mathbb{P}_e$ and $\rho\pi_2 = \mathbb{P}_f$ holds for the states $\rho = \mathbb{P}_{(\pi_1 e)\wedge(\pi_2 f)}$ with $e \in C$ and, furthermore, $(T^*\rho)\pi_1 = \mathbb{P}_e = (T^*\rho)\pi_2$ with $e \in C$ implies $T^*\rho = \mathbb{P}_{(\pi_1 e)\wedge(\pi_2 e)}$. Therefore by Lemma 1(b)

$$\mathbb{P}_{(\pi_1 e)\wedge(\pi_2 e)} = T^*\mathbb{P}_{(\pi_1 e)\wedge(\pi_2 f)} = \mathbb{P}_{T^{-1}((\pi_1 e)\wedge(\pi_2 f))}$$

and thus

$$T^{-1}\left((\pi_1 e) \wedge (\pi_2 f)\right) = \left((\pi_1 e) \wedge (\pi_2 e)\right)$$

for each $e \in C$. Now assume $e_1, e_2 \in C$ and consider

$$\mathbb{P}\left((\pi_1 e_2) \wedge (\pi_2 f)|(\pi_1 e_1) \wedge (\pi_2 f)\right).$$

On the one hand, the repeated application of Lemmas 1(a) and 2 yields

$$\begin{aligned}&\mathbb{P}\left((\pi_1 e_2) \wedge (\pi_2 f)|(\pi_1 e_1) \wedge (\pi_2 f)\right)\\ &\quad= \mathbb{P}\left(T^{-1}\left((\pi_1 e_2) \wedge (\pi_2 f)\right)|T^{-1}\left((\pi_1 e_1) \wedge (\pi_2 f)\right)\right)\\ &\quad= \mathbb{P}\left((\pi_1 e_2) \wedge (\pi_2 e_2)|(\pi_1 e_1) \wedge (\pi_2 e_1)\right)\\ &\quad= \mathbb{P}\left(\pi_1 e_2|\pi_1 e_1\right)\mathbb{P}\left(\pi_2 e_2|\pi_2 e_1\right)\\ &\quad= \mathbb{P}\left(e_2|e_1\right)\mathbb{P}\left(e_2|e_1\right)\\ &\quad= \left(\mathbb{P}\left(e_2|e_1\right)\right)^2\end{aligned}$$

and, on the other hand,

$$\begin{aligned}&\mathbb{P}\left((\pi_1 e_2) \wedge (\pi_2 f)|(\pi_1 e_1) \wedge (\pi_2 f)\right)\\ &\quad= \mathbb{P}\left(\pi_1 e_2|\pi_1 e_1\right)\mathbb{P}\left(\pi_2 f|\pi_2 f\right)\\ &\quad= \mathbb{P}\left(e_2|e_1\right)\mathbb{P}\left(f|f\right))\\ &\quad= \mathbb{P}\left(e_2|e_1\right).\end{aligned}$$

Therefore, $\left(\mathbb{P}\left(e_2|e_1\right)\right)^2 = \mathbb{P}\left(e_2|e_1\right)$ and $\mathbb{P}\left(e_2|e_1\right) \in \{0, 1\}$. This means that e_1 and e_2 are either orthogonal or identical.

If L is a finite Boolean algebra (i.e., classical), different atoms are orthogonal and a cloning transformation T is defined by extending to L the following permutation of the atoms in L: $T((\pi_1 e) \wedge (\pi_2 e)) = (\pi_1 e) \wedge (\pi_2 f)$ and $T((\pi_1 e) \wedge (\pi_2 f)) = (\pi_1 e) \wedge (\pi_2 e)$ for all $e \in C$, $Td = d$ for the other atoms d in L. However, non-orthogonal atoms are quite characteristic of quantum mechanics and Theorem 1 rules out that the corresponding atomic states can be cloned.

VI. QUANTUM MECHANICS

Quantum mechanics uses a special quantum logic; it consists of the self-adjoint projection operators on a Hilbert space H and is an orthomodular lattice. Compatibility here means that the self-adjoint projection operators commute. Conditions (F) and (H) in section III are satisfied, and the unique conditional probabilities exist (G) unless the dimension of H is two [21]. Moreover, it has been shown in Ref. [21] that, with two self-adjoint projection operators e and f on H, the conditional probability has the shape

$$\rho(f|e) = \frac{trace(aefe)}{trace(ae)} = \frac{trace(eaef)}{trace(ae)}$$

for a state ρ defined by the statistical operator a (i.e., a is a self-adjoint operator on H with non-negative spectrum and $trace(a) = 1$). The above identity reveals that conditionalization becomes identical with the state transition of the Lüders - von Neumann measurement process. Therefore, the conditional probabilities defined in section III can be regarded as a generalized mathematical model of projective quantum measurement.

$\mathbb{P}(f|e)$ exists with $\mathbb{P}(f|e) = s$ if and only if the operators e and f satisfy the algebraic identity $efe = se$. The state-independence of this conditional probability has not attracted much attention so far, although it may have profound implications for quantum foundations and interpretation, quantum information theory, and the philosophical question what constitutes physical reality. This topic needs further study; a first starting point is Ref. [23].

The atoms are the self-adjoint projections on the one-dimensional subspaces of H; if e is an atom and ξ a normalized vector in the corresponding one-dimensional subspace, then

$$\mathbb{P}(f|e) = \langle \xi | f\xi \rangle \,.$$

The atomic states thus coincide with the quantum mechanical pure states or vector states. Their general non-orthogonality is quite characteristic of quantum mechanics.

The quantum mechanical model of a composite system consisting of two copies is the Hilbert space tensor product $H \otimes H$. The self-adjoint projection operators e on H are mapped to two copies on $H \otimes H$ by $\pi_1(e) := e \otimes \mathbb{I}$ and $\pi_2(e) := \mathbb{I} \otimes e$. Note that (I) and (J) are then satisfied. The time evolution of the composite system is described by unitary transformations of $H \otimes H$ and therefore the cloning operation should be a unitary transformation. It defines an automorphism of the quantum logic of $H \otimes H$.

Theorem 1 thus includes the quantum mechanical no-cloning theorem for pure states as a special case. Instead of the Hilbert space and tensor product formalism, Theorem 1 requires only a few very basic principles; these are the existence and the uniqueness of the conditional probabilities and the existence of two compatible copies of the system in a larger system. Nevertheless, the proof of the no-cloning theorem in the quantum mechanical Hilbert space formalism can be mimicked, replacing the Hilbert space inner product $\langle \; | \; \rangle$ by the specific state-independent conditional probability $\mathbb{P}(\; | \;)$.

Theorem 1 considers only the atomic or pure states, while other approaches to a generalized no-cloning or no broadcasting theorem [1, 2, 20] include the mixed states. On the other hand, these approaches are restricted to finite-dimensional theories or universal cloning and need an explicit tensor product construction.

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